

On a space of entire functions and its Fourier transform

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ABSTRACT. A space of entire functions of several complex variables rapidly decreasing on \mathbb{R}^n and such that their growth along $i\mathbb{R}^n$ is majorized with the help of a family of weight functions is considered in this paper. For such space an equivalent description in terms of estimates on all of its partial derivatives as functions on \mathbb{R}^n and a Paley-Wiener type theorem are obtained.

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1 Introduction

1.1. On the problem. In the 1950's the study of W -type spaces started with the works of B.L. Gurevich [10], [11] and I.M. Gelfand and G.E. Shilov [8], [9]. They described them by means of the Fourier transform and then applied this description to study the uniqueness of the Cauchy problem of partial differential equations and their systems. These spaces generalize Gelfand-Shilov spaces of S -type [8]. So they are often called Gelfand-Shilov spaces of W -type.

Let us recall the definition of the Gelfand-Shilov spaces of W -type. Let M and Ω be differentiable functions on $[0, \infty)$ such that $M(0) = \Omega(0) = M'(0) = \Omega'(0) = 0$ and, moreover, so that their derivatives are continuous, increasing and unbounded at infinity. Considering \mathbb{R}^n with its usual norm $\|\cdot\|$,

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W_M is the space of all infinitely differentiable functions f on \mathbb{R}^n satisfying the following upper estimate on every partial derivative

$$|(D^\alpha f)(x)| \leq C_\alpha e^{-M(a\|x\|)}, \quad x \in \mathbb{R}^n,$$

for some positive constant a . Also, W^Ω is the space of entire functions f on \mathbb{C}^n satisfying the estimate

$$|\zeta^\alpha f(\zeta)| \leq C_\alpha e^{\Omega(b\|\eta\|)}, \quad \zeta = \xi + i\eta \in \mathbb{R}^n + i\mathbb{R}^n, \alpha \in \mathbb{Z}_+^n,$$

for some $b > 0$. And finally, W_M^Ω is the space of entire functions f on \mathbb{C}^n satisfying the estimate

$$|f(\xi + i\eta)| \leq C e^{-M(a\|\xi\|) + \Omega(b\|\eta\|)}, \quad \xi, \eta \in \mathbb{R}^n,$$

for some positive constants a, b and C .

W -type spaces and some their generalizations have been studied by many mathematicians. For example, new characterizations of W -type spaces and their generalizations were given by J. Chung, S.Y. Chung and D. Kim [4], [5], R.S. Pathak and S.K. Upadhyay [15], S.K. Upadhyay [17] (in terms of Fourier transform), N.G. De Bruijn [2], A.J.E.M. Janssen and S.J.L. van Eijndhoven [12], Jonggyu Cho [3] (by using the growth of their Wigner distributions). R.S. Pathak [14] and S.J.L. van Eijndhoven and M.J. Kerkhof [6] introduced new spaces of W -type and investigated the behaviour of the Hankel transform over them (see also [1]). New W -type spaces introduced by V.Ya. Fainberg, M.A. Soloviev [7] turned out to be useful for nonlocal theory of highly singular quantum fields.

In this paper we explore spaces of entire functions which are natural generalizations of W^Ω -type spaces. Namely, we work with the space $E(\Phi)$ of entire functions that we introduce next. Let $n \in \mathbb{N}$, $H(\mathbb{C}^n)$ be the space of entire functions on \mathbb{C}^n , $\|u\|$ be the Euclidean norm of $u \in \mathbb{R}^n(\mathbb{C}^n)$. Denote by $\mathcal{A}(\mathbb{R}^n)$ the set of all real-valued functions $g \in C(\mathbb{R}^n)$ satisfying the following conditions:

- 1) $g(x) = g(|x_1|, \dots, |x_n|)$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$;
- 2) the restriction of g to $[0, \infty)^n$ is nondecreasing in each variable;
- 3) $\lim_{x \rightarrow \infty} \frac{g(x)}{\|x\|} = +\infty$.

Let $\Phi = \{\varphi_\nu\}_{\nu=1}^\infty$ be a subset of $\mathcal{A}(\mathbb{R}^n)$ consisting of functions φ_ν satisfying the condition:

$i_0)$ for each $\nu \in \mathbb{N}$ and each $A > 0$ there exists a constant $C_{\nu,A} > 0$ such that

$$\varphi_\nu(x) + A \ln(1 + \|x\|) \leq \varphi_{\nu+1}(x) + C_{\nu,A}, \quad x \in \mathbb{R}^n.$$

For each $\nu \in \mathbb{N}$ and $m \in \mathbb{Z}_+$ consider the normed space

$$E_m(\varphi_\nu) = \{f \in H(\mathbb{C}^n) : p_{\nu,m}(f) = \sup_{z \in \mathbb{C}^n} \frac{|f(z)|(1 + \|z\|)^m}{e^{\varphi_\nu(I m z)}} < \infty\}.$$

Let $E(\varphi_\nu) = \bigcap_{m=0}^{\infty} E_m(\varphi_\nu)$. Obviously, $E_{m+1}(\varphi_\nu)$ is continuously embedded in $E_m(\varphi_\nu)$. Endow $E(\varphi_\nu)$ with a projective limit topology of spaces $E_m(\varphi_\nu)$. Note that if $f \in E(\varphi_\nu)$ then $p_{\nu+1,m}(f) \leq e^{C_{\nu,1}} p_{\nu,m}(f)$ for each $m \in \mathbb{Z}_+$. Hence, $E(\varphi_\nu)$ is continuously embedded in $E(\varphi_{\nu+1})$ for each $\nu \in \mathbb{N}$. Let $E(\Phi) = \bigcup_{\nu=1}^{\infty} E(\varphi_\nu)$. With the usual operations of addition and multiplication by complex numbers $E(\Phi)$ is a linear space. Supply $E(\Phi)$ with a topology of the inductive limit of spaces $E(\varphi_\nu)$.

In the paper we describe the space $E(\Phi)$ in terms of estimates on partial derivatives of functions on \mathbb{R}^n and study the Fourier transform of functions of $E(\Phi)$ under additional conditions on Φ . Results of the paper could be useful in harmonic analysis, theory of entire functions, convex analysis and in the study of partial differential and pseudo-differential operators.

1.2. Some notations. For $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in \mathbb{R}^n$ (\mathbb{C}^n) let denote $\langle u, v \rangle = u_1 v_1 + \dots + u_n v_n$.

For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ we follow the standard notations $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\alpha! = \alpha_1! \dots \alpha_n!$, $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$.

By s_n denote the surface area of the unit sphere in \mathbb{R}^n .

If $[0, \infty)^n \subseteq X \subset \mathbb{R}^n$ then for a function u on X denote by $u[e]$ the function defined by the rule: $u[e](x) = u(e^{x_1}, \dots, e^{x_n})$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

For an unbounded subset X of \mathbb{R}^n we denote by $\mathcal{B}(X)$ the set of all real-valued continuous functions g on X such that $\lim_{\substack{x \rightarrow \infty, \\ x \in X}} \frac{g(x)}{\|x\|} = +\infty$.

Let us recall that the Young-Fenchel conjugate of a function $g : \mathbb{R}^n \rightarrow [-\infty, +\infty]$ is the function $g^* : \mathbb{R}^n \rightarrow [-\infty, +\infty]$ defined by

$$g^*(x) = \sup_{y \in \mathbb{R}^n} (\langle x, y \rangle - g(y)), \quad x \in \mathbb{R}^n.$$

1.3. Main results and organization of the paper. Given a family $\Phi = \{\varphi_\nu\}_{\nu=1}^{\infty}$ as before and denoting $\varphi_\nu[e]$ by ψ_ν , consider the family $\Psi^* = \{\psi_\nu^*\}_{\nu=1}^{\infty}$. For each $\nu \in \mathbb{N}$ and $m \in \mathbb{Z}_+$ consider the normed space

$$\mathcal{E}_m(\psi_\nu^*) = \{f \in C^\infty(\mathbb{R}^n) : \rho_{m,\nu}(f) = \sup_{x \in \mathbb{R}^n, \alpha \in \mathbb{Z}_+^n} \frac{(1 + \|x\|)^m |(D^\alpha f)(x)|}{\alpha! e^{-\psi_\nu^*(\alpha)}} < \infty\}.$$

Let $\mathcal{E}(\psi_\nu^*) = \bigcap_{m=0}^{\infty} \mathcal{E}_m(\psi_\nu^*)$. Endow $\mathcal{E}(\psi_\nu^*)$ with the topology defined by the family of norms $\rho_{m,\nu}$ ($m \in \mathbb{Z}_+$). Let $\mathcal{E}(\Psi^*) = \bigcup_{\nu=1}^{\infty} \mathcal{E}(\psi_\nu^*)$. Supply $\mathcal{E}(\Psi^*)$ with an inductive limit topology of spaces $\mathcal{E}(\psi_\nu^*)$.

The first two theorems are aimed to characterize functions of the space $E(\Phi)$ in terms of estimates of their partial derivatives on \mathbb{R}^n .

Theorem 1. *For each $f \in E(\Phi)$ its restriction to \mathbb{R}^n belongs to $\mathcal{E}(\Psi^*)$.*

Theorem 2. *Let the family $\Phi = \{\varphi_\nu\}_{\nu=1}^{\infty}$ satisfies the additional conditions:
 $i_1)$ for each $\nu \in \mathbb{N}$ there exist constants $\sigma_\nu > 1$ and $\gamma_\nu > 0$ such that*

$$\varphi_\nu(\sigma_\nu x) \leq \varphi_{\nu+1}(x) + \gamma_\nu, \quad x \in \mathbb{R}^n;$$

$i_2)$ for each $\nu \in \mathbb{N}$ there exists a constant $K_\nu > 0$ such that

$$\varphi_\nu(x + \xi) \leq \varphi_{\nu+1}(x) + K_\nu, \quad x \in [0, \infty)^n, \xi \in [0, 1]^n.$$

Then each function $f \in \mathcal{E}(\Psi^)$ admits an (unique) extension to entire function belonging to $E(\Phi)$.*

The proofs of these theorems are given in section 3 by standard techniques. These proofs allow us to obtain additional information on the structure of the space $E(\Phi)$ (see Proposition 2).

Section 4 is devoted to description of the space $E(\Phi)$ in terms of Fourier transform under additional conditions on Φ . For each $\nu \in \mathbb{N}$ and $m \in \mathbb{Z}_+$ define the normed space

$$G_m(\psi_\nu^*) = \{f \in C^m(\mathbb{R}^n) : \|f\|_{m,\psi_\nu^*} = \sup_{x \in \mathbb{R}^n, |\alpha| \leq m, \beta \in \mathbb{Z}_+^n} \frac{|x^\beta (D^\alpha f)(x)|}{\beta! e^{-\psi_\nu^*(\beta)}} < \infty\}.$$

Let $G(\psi_\nu^*) = \bigcap_{m=0}^{\infty} G_m(\psi_\nu^*)$. Endow $G(\psi_\nu^*)$ with the topology defined by the family of norms $\|\cdot\|_{m,\psi_\nu^*}$ ($m \in \mathbb{Z}_+$). Considering $G(\Psi^*) = \bigcup_{\nu=1}^{\infty} G(\psi_\nu^*)$ supply it with the topology of the inductive limit of spaces $G(\psi_\nu^*)$.

For $f \in E(\Phi)$ define its Fourier transform \hat{f} by the formula

$$\hat{f}(x) = \int_{\mathbb{R}^n} f(\xi) e^{-i\langle x, \xi \rangle} d\xi, \quad x \in \mathbb{R}^n.$$

Theorem 3. *Let the family $\Phi = \{\varphi_\nu\}_{\nu=1}^{\infty}$ satisfies the condition $i_2)$ of Theorem 2 and the following two conditions:*

i_3) for each $\nu \in \mathbb{N}$ there is a constant $a_\nu > 0$ such that

$$\varphi_\nu(2x) \leq \varphi_{\nu+1}(x) + a_\nu, \quad x \in \mathbb{R}^n;$$

i_4) for each $\nu \in \mathbb{N}$ there is a constant $l_\nu > 0$ such that

$$2\varphi_\nu(x) \leq \varphi_{\nu+1}(x) + l_\nu, \quad x \in \mathbb{R}^n.$$

Then Fourier transform $\mathcal{F} : f \in E(\Phi) \rightarrow \hat{f}$ establishes an isomorphism between the spaces $E(\Phi)$ and $G(\Psi^*)$.

Moreover, let $\Phi^* = \{\varphi_\nu^*\}_{\nu=1}^\infty$. For each $\nu \in \mathbb{N}$ and $m \in \mathbb{Z}_+$ define the space

$$GS_m(\varphi_\nu^*) = \{f \in C^m(\mathbb{R}^n) : q_{m,\nu}(f) = \sup_{\substack{x \in \mathbb{R}^n, \\ |\alpha| \leq m}} \frac{|(D^\alpha f)(x)|}{e^{-\varphi_\nu^*(x)}} < \infty\}.$$

For each $\nu \in \mathbb{N}$ let $GS(\varphi_\nu^*) = \bigcap_{m \in \mathbb{Z}_+} GS_m(\varphi_\nu^*)$. Endow $GS(\varphi_\nu^*)$ with the topology defined by the family of norms $q_{m,\nu}$ ($m \in \mathbb{Z}_+$). Let $GS(\Phi^*) = \bigcup_{\nu \in \mathbb{N}} GS(\varphi_\nu^*)$. Supply $GS(\Phi^*)$ with an inductive limit topology of spaces $GS(\varphi_\nu^*)$.

The main result of the section 5 is the following theorem.

Theorem 4. *Let functions of the family Φ be convex and satisfy the condition i_2) of Theorem 2 and the condition i_3) of Theorem 3. Then $G(\Psi^*) = GS(\Phi^*)$.*

The proof of Theorem 4 is essentially based on results of subsection 5.1 where some properties of the Young-Fenchel transform are considered.

Remark 1. Note that if functions φ_m are defined on \mathbb{R}^n by the formula $\varphi_m(x) = \Omega(2^m \|x\|)$ then the family Φ satisfies the assumptions of Theorem 4.

2 Auxiliary results

In the proofs of the main results the following four lemmas will be useful.

Lemma 1. *Let $g \in \mathcal{B}([0, \infty)^n)$. Then for each $M > 0$ there exists a constant $A > 0$ such that*

$$(g[e])^*(x) \leq \sum_{1 \leq j \leq n: x_j \neq 0} (x_j \ln \frac{x_j}{M} - x_j) + A, \quad x \in [0, \infty)^n.$$

Proof. For each $M > 0$ we can find a number $A > 0$ such that for all $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ we have $g[e](y) \geq \sum_{j=1}^n M e^{y_j} - A$. Hence, for $x = (x_1, \dots, x_n) \in [0, \infty)^n$

$$\begin{aligned} (g[e])^*(x) &= \sup_{y \in \mathbb{R}^n} (\langle x, y \rangle - g[e](y)) \leq \sup_{(y_1, \dots, y_n) \in \mathbb{R}^n} \sum_{j=1}^n (x_j y_j - M e^{y_j}) + A \\ &= \sum_{1 \leq j \leq n: x_j \neq 0} \sup_{y_j \in \mathbb{R}} (x_j y_j - M e^{y_j}) + A = \sum_{1 \leq j \leq n: x_j \neq 0} (x_j \ln \frac{x_j}{M} - x_j) + A. \quad \square \end{aligned}$$

Corollary 1. Let $g \in \mathcal{B}([0, \infty)^n)$. Then for each $b > 0$ the series $\sum_{|j| \geq 0} \frac{e^{(g[e])^*(j)}}{b^{|j|} j!}$ and $\sum_{|j| \geq 0} \frac{e^{(g[e])^*(j)}}{b^{|j|} |j|!}$ are converging.

Remark 2. Note that if $g \in \mathcal{B}([0, \infty)^n)$ then it is easy to see that $(g[e])^*(x) = +\infty$ for $x \notin [0, \infty)^n$, $(g[e])^*(x) > -\infty$ for $x \in \mathbb{R}^n$. Also notice that $\lim_{\substack{x \rightarrow \infty, \\ x \in [0, \infty)^n}} \frac{(g[e])^*(x)}{\|x\|} = +\infty$. Indeed, from the definition of $(g[e])^*$ it follows that $(g[e])^*(x) \geq \langle x, t \rangle - (g[e])(t)$ for all $x \in [0, \infty)^n$ and $t \in \mathbb{R}^n$. So if positive M is arbitrary then from this inequality we get that $(g[e])^*(x) \geq M\|x\| - g[e](\frac{Mx}{\|x\|})$ for $x \neq 0$. From this our assertion follows.

Lemma 2. Let $u, v \in \mathcal{B}([0, \infty)^n)$ and for some $l > 0$

$$2u(x) \leq v(x) + l, \quad x \in [0, \infty)^n.$$

Then

$$(v[e])^*(x + y) \leq (u[e])^*(x) + (u[e])^*(y) + l, \quad x, y \in [0, \infty)^n.$$

Proof. Let $x, y \in [0, \infty)^n$. Then for each $t \in \mathbb{R}^n$ we have that

$$(u[e])^*(x) + (u[e])^*(y) \geq \langle x + y, t \rangle - 2u[e](t) \geq \langle x + y, t \rangle - v[e](t) - l.$$

From this it follows that

$$(v[e])^*(x + y) = \sup_{t \in \mathbb{R}^n} (\langle x + y, t \rangle - v[e](t)) \leq (u[e])^*(x) + (u[e])^*(y) + l. \quad \square$$

Lemma 3. Let $u, v \in \mathcal{B}([0, \infty)^n)$ and there are constants $\sigma > 1$ and $\gamma > 0$ such that

$$u(\sigma x) \leq v(x) + \gamma, \quad x \in [0, \infty)^n.$$

Then for $x = (x_1, \dots, x_n) \in [0, \infty)^n$ one has

$$(u[e])^*(x) - (v[e])^*(x) \geq \sum_{j=1}^n x_j \ln \sigma - \gamma.$$

Proof. Note that by Lemma 1 and Remark 2 $(u[e])^*(x) < \infty$ and $(v[e])^*(x) < \infty$ for $x \in [0, \infty)^n$. From the condition on u and v it follows that

$$u[e](t + \eta) \leq v[e](t) + \gamma, \quad t \in \mathbb{R}^n,$$

where $\eta = (\ln \sigma, \dots, \ln \sigma)$. Then for $x = (x_1, \dots, x_n) \in [0, \infty)^n$

$$\begin{aligned} (u[e])^*(x) - (v[e])^*(x) &= \sup_{t \in \mathbb{R}^n} (\langle x, t \rangle - u[e](t)) - \sup_{t \in \mathbb{R}^n} (\langle x, t \rangle - v[e](t)) \\ &\geq \sup_{t \in \mathbb{R}^n} (\langle x, t \rangle - u[e](t)) - \sup_{t \in \mathbb{R}^n} (\langle x, t \rangle - u[e](t + \eta)) - \gamma \\ &= \sup_{t \in \mathbb{R}^n} (\langle x, t \rangle - u[e](t)) - \sup_{t \in \mathbb{R}^n} (\langle x, t + \eta \rangle - u[e](t + \eta)) + \langle x, \eta \rangle - \gamma = \sum_{j=1}^n x_j \ln \sigma - \gamma. \end{aligned}$$

Lemma 4. Let $g = (g_1, \dots, g_n)$ be a vector-function on \mathbb{R}^n with convex components $g_j : \mathbb{R}^n \rightarrow [0, \infty)$ and a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be such that $f|_{[0, \infty)^n}$ is convex and nondecreasing in each argument. Then $f \circ g$ is convex on \mathbb{R}^n .

Proof. Let $x, y \in \mathbb{R}^n$, $\alpha \in [0, 1]$. Then for each $j = 1, \dots, n$ we have that $0 \leq g_j(\alpha x + (1 - \alpha)y) \leq \alpha g_j(x) + (1 - \alpha)g_j(y)$. From this using monotonicity of f on $[0, \infty)^n$ we get that $f(g(\alpha x + (1 - \alpha)y)) \leq f(\alpha g(x) + (1 - \alpha)g(y))$. Now using the convexity of f on $[0, \infty)^n$ we obtain the required relation $f(g(\alpha x + (1 - \alpha)y)) \leq \alpha f(g(x)) + (1 - \alpha)f(g(y))$. \square

3 Equivalent description of the space $E(\Phi)$

3.1. Proof of Theorem 1. Let $f \in E(\Phi)$. Then $f \in E(\varphi_\nu)$ for some $\nu \in \mathbb{N}$. Let $m \in \mathbb{Z}_+$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ be arbitrary. For $j = 1, \dots, n$ let R_j be an arbitrary positive number. For $R = (R_1, \dots, R_n)$ let $L_R(x) = \{\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n : |\zeta_j - x_j| = R_j, j = 1, \dots, n\}$. Using Cauchy integral formula we have that

$$(1 + \|x\|)^m (D^\alpha f)(x) = \frac{\alpha!}{(2\pi i)^n} \int_{L_R(x)} \frac{f(\zeta)(1 + \|x\|)^m d\zeta}{(\zeta_1 - x_1)^{\alpha_1+1} \dots (\zeta_n - x_n)^{\alpha_n+1}}.$$

From this we get that

$$\begin{aligned} (1 + \|x\|)^m |(D^\alpha f)(x)| &\leq \frac{\alpha!}{(2\pi)^n} \int_{L_R(x)} \frac{(1 + \|x - \zeta\|)^m (1 + \|\zeta\|)^m |f(\zeta)| |d\zeta|}{|\zeta_1 - x_1|^{\alpha_1+1} \dots |\zeta_n - x_n|^{\alpha_n+1}} \\ &\leq \frac{\alpha! p_{\nu, m}(f)(1 + \|R\|)^m e^{\varphi_\nu(R)}}{R^\alpha}. \end{aligned}$$

Using the condition i_0) on Φ we obtain that

$$(1 + \|x\|)^m |(D^\alpha f)(x)| \leq e^{C_{\nu,m}} \alpha! p_{\nu,m}(f) \frac{e^{\varphi_{\nu+1}(R)}}{R^\alpha}.$$

Hence,

$$\begin{aligned} (1 + \|x\|)^m |(D^\alpha f)(x)| &\leq e^{C_{\nu,m}} \alpha! p_{\nu,m}(f) \inf_{R \in (0, \infty)^n} \frac{e^{\varphi_{\nu+1}(R)}}{R^\alpha} \\ &= e^{C_{\nu,m}} \alpha! p_{\nu,m}(f) \exp(-\sup_{r \in \mathbb{R}^n} (\langle \alpha, r \rangle - \psi_{\nu+1}(r))) = e^{C_{\nu,m}} \alpha! p_{\nu,m}(f) e^{-\psi_{\nu+1}^*(\alpha)}. \end{aligned}$$

From this it follows that for each $m \in \mathbb{Z}_+$

$$\rho_{m,\nu+1}(f|_{\mathbb{R}^n}) \leq e^{C_{\nu,m}} p_{\nu,m}(f). \quad (1)$$

Therefore, $f|_{\mathbb{R}^n} \in \mathcal{E}(\psi_{\nu+1}^*)$. Thus, $f|_{\mathbb{R}^n} \in \mathcal{E}(\Psi^*)$. Note that the inequality (1) ensures the continuity of the embedding. \square

Proof of Theorem 2. Let $f \in \mathcal{E}(\Psi^*)$. Then $f \in \mathcal{E}(\psi_\nu^*)$ for some $\nu \in \mathbb{N}$. Hence, for each $m \in \mathbb{Z}_+$ we have that

$$(1 + \|x\|)^m |(D^\alpha f)(x)| \leq \rho_{m,\nu}(f) \alpha! e^{-\psi_\nu^*(\alpha)}, \quad x \in \mathbb{R}^n, \alpha \in \mathbb{Z}_+^n. \quad (2)$$

Since $\lim_{\substack{x \rightarrow \infty, \\ x \in \Pi_n}} \frac{\psi_\nu^*(x)}{\|x\|} = +\infty$ (see Remark 2) then for each $\varepsilon > 0$ there is a constant $c_\varepsilon > 0$ such that for all $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{Z}_+^n$ we have that

$$|(D^\alpha f)(x)| \leq c_\varepsilon \varepsilon^{|\alpha|} \alpha!. \quad (3)$$

Hence, the sequence $(\sum_{|\alpha| \leq k} \frac{(D^\alpha f)(0)}{\alpha!} x^\alpha)_{k=1}^\infty$ converges to f uniformly on

compacts of \mathbb{R}^n . Also from (3) it follows that the series $\sum_{|\alpha| \geq 0} \frac{(D^\alpha f)(0)}{\alpha!} z^\alpha$

converges uniformly on compacts of \mathbb{C}^n and, hence, its sum $F_f(z)$ is an entire function. Obviously, $F_f(x) = f(x)$, $x \in \mathbb{R}^n$. The uniqueness of holomorphic continuation is obvious.

Now we have to show that $F_f \in E(\Phi)$. We will estimate a growth of F_f using the inequality (2) and the Taylor series expansion of $F_f(z)$ ($z = x + iy$, $x, y \in \mathbb{R}^n$) with respect to a point x :

$$F_f(z) = \sum_{|\alpha| \geq 0} \frac{(D^\alpha f)(x)}{\alpha!} (iy)^\alpha.$$

Let $m \in \mathbb{Z}_+$ be arbitrary. Then

$$\begin{aligned}
(1 + \|z\|)^m |F_f(z)| &\leq \sum_{|\alpha| \geq 0} \frac{1}{\alpha!} (1 + \|x\|)^m (1 + \|y\|)^m \prod_{j=1}^n (1 + |y_j|)^{\alpha_j} |(D^\alpha f)(x)| \\
&\leq \rho_{m,\nu}(f) (1 + \|y\|)^m \sum_{|\alpha| \geq 0} e^{-\psi_\nu^*(\alpha)} \prod_{j=1}^n (1 + |y_j|)^{\alpha_j} \\
&\leq \rho_{m,\nu}(f) (1 + \|y\|)^m \sum_{|\alpha| \geq 0} \frac{\prod_{j=1}^n (1 + |y_j|)^{\alpha_j}}{e^{\psi_{\nu+1}^*(\alpha)}} e^{\psi_{\nu+1}^*(\alpha) - \psi_\nu^*(\alpha)}.
\end{aligned}$$

Since Φ satisfies the condition i_1) then by Lemma 3

$$\psi_\nu^*(x) - \psi_{\nu+1}^*(x) \geq \delta_\nu \sum_{j=1}^n x_j - \gamma_\nu, \quad x = (x_1, \dots, x_n) \in [0, \infty)^n,$$

where $\delta_\nu = \ln \sigma_\nu$. Using this inequality and denoting the sum of the series $\sum_{|\alpha| \geq 0} e^{\psi_{\nu+1}^*(\alpha) - \psi_\nu^*(\alpha)}$ by B_ν we have that

$$\begin{aligned}
(1 + \|z\|)^m |F_f(z)| &\leq B_\nu \rho_{m,\nu}(f) (1 + \|y\|)^m \sup_{|\alpha| \geq 0} \frac{\prod_{j=1}^n (1 + |y_j|)^{\alpha_j}}{e^{\psi_{\nu+1}^*(\alpha)}} \\
&\leq B_\nu \rho_{m,\nu}(f) (1 + \|y\|)^m e^{\sup_{(t_1, \dots, t_n) \in [0, \infty)^n} (t_1 \ln(1 + |y_1|) + \dots + t_n \ln(1 + |y_n|) - \psi_{\nu+1}^*(t))}.
\end{aligned}$$

Taking into account Remark 2 we obtain that

$$(1 + \|z\|)^m |F_f(z)| \leq B_\nu \rho_{m,\nu}(f) e^{(\psi_{\nu+1}^*)^*(\ln(1 + |y_1|), \dots, \ln(1 + |y_n|)) + m \ln(1 + \|y\|)}. \quad (4)$$

Note that for each $k \in \mathbb{N}$ and $A > 0$ from condition i_0) we can find a constant $C_1(k, A) > 0$ such that for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$$\psi_k(x) + A \sum_{j=1}^n x_j \leq \psi_{k+1}(x) + C_1(k, A).$$

So for all $\xi \in [0, \infty)^n$ and $A > 0$

$$\psi_k^*(\xi) = \sup_{x \in \mathbb{R}^n} (\langle \xi, x \rangle - \psi_k(x)) \geq \sup_{x \in \mathbb{R}^n} (\langle \xi, x \rangle - \psi_{k+1}(x) + A \sum_{j=1}^n x_j) - C_1(k, A).$$

Put $\Lambda := (A, \dots, A) \in \mathbb{R}^n$. Then from the previous estimate we have that

$$\psi_k^*(\xi) \geq \sup_{x \in \mathbb{R}^n} (\langle \xi + \Lambda, x \rangle - \psi_{k+1}(x)) - C_1(k, A) = \psi_{k+1}^*(\xi + \Lambda) - C_1(k, A).$$

Further, for all $x \in [0, \infty)^n$ and $A > 0$

$$\begin{aligned} (\psi_k^*)^*(x) &= \sup_{\xi \in \mathbb{R}^n} (\langle x, \xi \rangle - \psi_k^*(\xi)) = \sup_{\xi \in [0, \infty)^n} (\langle x, \xi \rangle - \psi_k^*(\xi)) \\ &\leq \sup_{\xi \in [0, \infty)^n} (\langle x, \xi \rangle - \psi_{k+1}^*(\xi + \Lambda)) + C_1(k, A) \\ &= \sup_{\xi \in [0, \infty)^n} (\langle x, \xi + \Lambda \rangle - \psi_{k+1}^*(\xi + \Lambda)) - A \sum_{j=1}^n x_j + C_1(k, A) \\ &\leq \sup_{\xi \in [0, \infty)^n} (\langle x, \xi \rangle - \psi_{k+1}^*(\xi)) - A \sum_{j=1}^n x_j + C_1(k, A) \\ &= (\psi_{k+1}^*)^*(x) - A \sum_{j=1}^n x_j + C_1(k, A). \end{aligned}$$

Thus, for each $k \in \mathbb{N}$ and $A > 0$ we have

$$(\psi_k^*)^*(x) + A \sum_{j=1}^n x_j \leq (\psi_{k+1}^*)^*(x) + C_1(k, A), \quad x \in [0, \infty)^n. \quad (5)$$

Now using the inequality (5) for $A = m$ and $k = \nu + 1$, we obtain from the estimate (4) that

$$(1 + \|z\|)^m |F_f(z)| \leq B_\nu \rho_{m, \nu}(f) e^{C_1(\nu+1, m)} e^{(\psi_{\nu+2}^*)^*(\ln(1+|y_1|), \dots, \ln(1+|y_n|))}. \quad (6)$$

Since $(\psi_{\nu+2}^*)^*(t) \leq \psi_{\nu+2}(t)$ for $t \in [0, \infty)^n$ then from (6) we get that

$$(1 + \|z\|)^m |F_f(z)| \leq B_\nu \rho_{m, \nu}(f) e^{C_1(\nu+1, m)} e^{\psi_{\nu+2}(\ln(1+|y_1|), \dots, \ln(1+|y_n|))}.$$

In other words,

$$(1 + \|z\|)^m |F_f(z)| \leq B_\nu \rho_{m, \nu}(f) e^{C_1(\nu+1, m)} e^{\varphi_{\nu+2}(1+|y_1|, \dots, 1+|y_n|)}.$$

Using the condition i_2) on Φ it is possible to find a constant $K_{\nu, m} > 0$ such that for all $z \in \mathbb{C}^n$

$$(1 + \|z\|)^m |F_f(z)| \leq K_{\nu, m} \rho_{m, \nu}(f) e^{\varphi_{\nu+3}(|Im z_1|, \dots, |Im z_n|)}. \quad (7)$$

Thus, for each $m \in \mathbb{Z}_+$ we have that $p_{\nu+3,m}(F_f) \leq K_{\nu,m}\rho_{m,\nu}(f)$. Hence, $F_f \in E(\varphi_{\nu+3})$. Thus, $F_f \in E(\Phi)$. Also note that the last inequality ensures the continuity of the embedding. \square

3.2. Another structure of $E(\Phi)$. For each $\nu \in \mathbb{N}$ and $m \in \mathbb{Z}_+$ consider the normed space

$$\mathcal{H}_m(\varphi_\nu) = \{f \in H(\mathbb{C}^n) : \sigma_{\nu,m}(f) = \sup_{z \in \mathbb{C}^n} \frac{|f(z)|(1 + \|z\|)^m}{e^{(\psi_\nu^*)^*(\ln(1+|Imz_1|), \dots, \ln(1+|Imz_n|))}} < \infty\}.$$

Let $\mathcal{H}(\varphi_\nu) = \bigcap_{m=0}^{\infty} \mathcal{H}_m(\varphi_\nu)$. Obviously, $\mathcal{H}_{m+1}(\varphi_\nu)$ is continuously embedded in $\mathcal{H}_m(\varphi_\nu)$. Endow $\mathcal{H}(\varphi_\nu)$ with a projective limit topology of spaces $\mathcal{H}_m(\varphi_\nu)$. Note that if $f \in \mathcal{H}(\varphi_\nu)$ then using the inequality (5) we have that $\sigma_{\nu+1,m}(f) \leq e^{C_1(\nu,1)}\sigma_{\nu,m}(f)$ for each $m \in \mathbb{Z}_+$. Thus, $\mathcal{H}(\varphi_\nu)$ is continuously embedded in $\mathcal{H}(\varphi_{\nu+1})$ for each $\nu \in \mathbb{N}$. Supply $\mathcal{H}(\Phi) = \bigcup_{\nu=1}^{\infty} \mathcal{H}(\varphi_\nu)$ with the topology of the inductive limit of spaces $\mathcal{H}(\varphi_\nu)$.

Proposition 1. *Let all the functions of the family Φ satisfy the condition i_2) of Theorem 2 and every function ψ_ν be convex on \mathbb{R}^n ($\nu \in \mathbb{N}$). Then $E(\Phi) = \mathcal{H}(\Phi)$.*

Proof. By assumption each function ψ_ν is convex and continuous on \mathbb{R}^n . Since the Young-Fenchel conjugation is involutive (see [16], Theorem 12.2) it follows that $(\psi_\nu^*)^* = \psi_\nu$. Thus, for each $\nu \in \mathbb{N}$ and $t = (t_1, \dots, t_n) \in [0, \infty)^n$ we have that

$$\begin{aligned} (\psi_\nu^*)^*(\ln(1+t_1), \dots, \ln(1+t_n)) &= \psi_\nu(\ln(1+t_1), \dots, \ln(1+t_n)) \\ &= \varphi_\nu(1+t_1, \dots, 1+t_n). \end{aligned}$$

From this, and taking into account that functions of the family Φ are nondecreasing in each variable in $[0, \infty)^n$, we get that

$$(\psi_\nu^*)^*(\ln(1+t_1), \dots, \ln(1+t_n)) \geq \varphi_\nu(t), \quad t = (t_1, \dots, t_n) \in [0, \infty)^n.$$

On the other hand using the condition i_2) of Theorem 2 we have

$$(\psi_\nu^*)^*(\ln(1+t_1), \dots, \ln(1+t_n)) \leq \varphi_{\nu+1}(t) + K_\nu, \quad t \in [0, \infty)^n. \quad (8)$$

From these inequalities the assertion follows. \square

Using Theorems 1 and 2 we can prove the following

Proposition 2. *Let the family Φ satisfies the conditions of Theorem 2. Then $E(\Phi) = \mathcal{H}(\Phi)$.*

Proof. Thanks to the condition i_2) the inequality (8) holds. Using it we have that for each $m \in \mathbb{Z}_+$

$$p_{\nu+1,m}(f) \leq e^{K_\nu} \sigma_{\nu,m}(f), \quad f \in \mathcal{H}(\varphi_\nu).$$

Hence, the identity embedding $I : \mathcal{H}(\Phi) \rightarrow E(\Phi)$ is continuous.

The mapping I is surjective too. Indeed, if $f \in E(\Phi)$ then $f \in E(\varphi_\nu)$ for some $\nu \in \mathbb{N}$. Let $m \in \mathbb{Z}_+$ be arbitrary. By the inequality (1) we have that $\rho_{m,\nu+1}(f|_{\mathbb{R}^n}) \leq e^{C_{\nu,m}} p_{\nu,m}(f)$. From this and the inequality (6) (with ν replaced by $\nu + 1$) we obtain that

$$\sigma_{\nu+3,m}(f) \leq A_{\nu,m} p_{\nu,m}(f),$$

where $A_{\nu,m}$ is some positive constant. Hence, $f \in \mathcal{H}(\varphi_{\nu+3})$. So, $f \in \mathcal{H}(\Phi)$. Moreover, the last estimate shows that the inverse mapping I^{-1} is continuous. Hence, the equality $E(\Phi) = \mathcal{H}(\Phi)$ is topological too. \square

4 Fourier transform of $E(\Phi)$

Recall two notations which will be used in the proof of Theorem 3. Namely, for $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$ the notation $\alpha \leq \beta$ indicates that $\alpha_j \leq \beta_j$ ($j = 1, 2, \dots, n$) and in such case $\binom{\beta}{\alpha} := \prod_{j=1}^n \binom{\beta_j}{\alpha_j}$, where $\binom{\beta_j}{\alpha_j}$ are the binomial coefficients.

Proof of Theorem 3. Let $\nu \in \mathbb{N}$ and $f \in E(\varphi_\nu)$. Let $\alpha, \beta \in \mathbb{Z}_+^n$, $x, \eta \in \mathbb{R}^n$. Then

$$x^\beta (D^\alpha \hat{f})(x) = x^\beta \int_{\mathbb{R}^n} f(\zeta) (-i\zeta)^\alpha e^{-i\langle x, \zeta \rangle} d\xi, \quad \zeta = \xi + i\eta.$$

From this equality we have that

$$\begin{aligned} |x^\beta (D^\alpha \hat{f})(x)| &\leq \int_{\mathbb{R}^n} |f(\zeta)| \|\zeta\|^{|\alpha|} e^{\langle x, \eta \rangle} \prod_{j=1}^n |x_j|^{\beta_j} d\xi \\ &\leq \int_{\mathbb{R}^n} |f(\zeta)| (1 + \|\zeta\|)^{n+|\alpha|+1} e^{\langle x, \eta \rangle} \prod_{j=1}^n |x_j|^{\beta_j} \frac{d\xi}{(1 + \|\xi\|)^{n+1}}. \end{aligned}$$

Hence,

$$|x^\beta (D^\alpha \hat{f})(x)| \leq s_n p_{\nu, n+|\alpha|+1}(f) e^{\varphi_\nu(\eta)} e^{\langle x, \eta \rangle} \prod_{j=1}^n |x_j|^{\beta_j}. \quad (9)$$

If $|\beta| = 0$ then (putting $\eta = 0$ in (9)) we have that

$$|(D^\alpha \hat{f})(x)| \leq s_n e^{\varphi_\nu(0)} p_{\nu, n+|\alpha|+1}(f). \quad (10)$$

Now let consider the case when $|\beta| > 0$. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ let $\theta(x)$ be a point in \mathbb{R}^n with coordinates θ_j defined as follows: $\theta_j = \frac{x_j}{|x_j|}$ if $x_j \neq 0$ and $\theta_j = 0$ if $x_j = 0$ ($j = 1, \dots, n$). Let $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ have strictly positive coordinates. Put $\eta = -(\theta_1 t_1, \dots, \theta_n t_n)$. Then from (9) we get that

$$\begin{aligned} |x^\beta (D^\alpha \hat{f})(x)| &\leq s_n p_{\nu, n+|\alpha|+1}(f) e^{\varphi_\nu(t)} \prod_{j \in \{1, \dots, n\}: \beta_j \neq 0} \frac{|x_j|^{\beta_j}}{e^{t_j |x_j|}} \\ &\leq s_n p_{\nu, n+|\alpha|+1}(f) e^{\varphi_\nu(t)} \prod_{j \in \{1, \dots, n\}: \beta_j \neq 0} \sup_{r_j > 0} \frac{r_j^{\beta_j}}{e^{t_j r_j}} \\ &= s_n p_{\nu, n+|\alpha|+1}(f) \exp(\varphi_\nu(t) + \sum_{1 \leq j \leq n: \beta_j \neq 0} \sup_{r_j > 0} (-t_j r_j + \beta_j \ln r_j)) \\ &= s_n p_{\nu, n+|\alpha|+1}(f) \exp(\varphi_\nu(t) + \sum_{1 \leq j \leq n: \beta_j \neq 0} (\beta_j \ln \beta_j - \beta_j) - \sum_{j=1}^n \beta_j \ln t_j). \end{aligned}$$

From this we have that

$$|x^\beta (D^\alpha \hat{f})(x)| \leq s_n p_{\nu, n+|\alpha|+1}(f) e^{\sum_{1 \leq j \leq n: \beta_j \neq 0} \beta_j \ln \frac{\beta_j}{e} + \inf_{t=(t_1, \dots, t_n) \in (0, \infty)^n} (\varphi_\nu(t) - \sum_{j=1}^n \beta_j \ln t_j)}.$$

Note that for each $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}_+^n$

$$\begin{aligned} &\inf_{t=(t_1, \dots, t_n) \in (0, \infty)^n} (-\mu_1 \ln t_1 - \dots - \mu_n \ln t_n + \varphi_\nu(t)) \\ &= -\sup_{u \in \mathbb{R}^n} (\langle \mu, u \rangle - \psi_\nu(u)) = -\psi_\nu^*(\mu), \end{aligned}$$

From this and the previous estimate it follows that

$$|x^\beta (D^\alpha \hat{f})(x)| \leq s_n p_{\nu, n+|\alpha|+1}(f) e^{\sum_{1 \leq j \leq n: \beta_j \neq 0} \beta_j \ln \frac{\beta_j}{e}} e^{-\psi_\nu^*(\beta)}.$$

Now from this and (10) we have that for $\alpha, \beta \in \mathbb{Z}_+^n$, $x \in \mathbb{R}^n$

$$|x^\beta (D^\alpha \hat{f})(x)| \leq s_n p_{\nu, n+|\alpha|+1}(f) \beta! e^{-\psi_\nu^*(\beta)}.$$

From this it follows that for each $m \in \mathbb{Z}_+$

$$\max_{|\alpha| \leq m} \sup_{x \in \mathbb{R}^n, \beta \in \mathbb{Z}_+^n} \frac{|x^\beta (D^\alpha \hat{f})(x)|}{\beta! e^{-\psi_\nu^*(\beta)}} \leq s_n p_{\nu, n+m+1}(f), \quad f \in E(\varphi_\nu).$$

In other words, for each $m \in \mathbb{Z}_+$

$$\|\hat{f}\|_{m, \psi_\nu^*} \leq s_n p_{\nu, n+m+1}(f), \quad f \in E(\varphi_\nu).$$

From this inequality it follows that the linear mapping $\mathcal{F} : f \in E(\Phi) \rightarrow \hat{f}$ acts from $E(\Phi)$ to $G(\Psi^*)$ and is continuous.

Let us show that \mathcal{F} is surjective. Take $g \in G(\Psi^*)$. Then $g \in G(\psi_\nu^*)$ for some $\nu \in \mathbb{N}$. Let $m \in \mathbb{Z}_+$ be arbitrary. Then for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq m$, $\beta \in \mathbb{Z}_+^n$, $x \in \mathbb{R}^n$ we have that

$$|x^\beta (D^\alpha g)(x)| \leq \|g\|_{m, \psi_\nu^*} \beta! e^{-\psi_\nu^*(\beta)}.$$

Using this inequality and the equality

$$\begin{aligned} |(D^\alpha g)(x)| \prod_{k=1}^n (1 + |x_k|)^{\beta_k} &= |(D^\alpha g)(x)| \prod_{k=1}^n \sum_{j_k=0}^{\beta_k} \binom{\beta_k}{j_k} |x_k|^{j_k} \\ &= \sum_{\substack{j \in \mathbb{Z}_+^n : \\ (0, \dots, 0) \leq j \leq \beta}} \binom{\beta}{j} |x^j (D^\alpha g)(x)| \end{aligned}$$

we have that

$$|(D^\alpha g)(x)| \prod_{k=1}^n (1 + |x_k|)^{\beta_k} \leq \|g\|_{m, \psi_\nu^*} \sum_{\substack{j \in \mathbb{Z}_+^n : \\ (0, \dots, 0) \leq j \leq \beta}} \binom{\beta}{j} j! e^{-\psi_\nu^*(j)}. \quad (11)$$

Now note that since the family Φ satisfies the condition $i_4)$ then with the help of Lemma 2 we have that for each $k \in \mathbb{N}$

$$\psi_{k+1}^*(x + y) \leq \psi_k^*(x) + \psi_k^*(y) + l_k, \quad x, y \in [0, \infty)^n. \quad (12)$$

Using this inequality we have from (11) that

$$|(D^\alpha g)(x)| \prod_{k=1}^n (1 + |x_k|)^{\beta_k} \leq \|g\|_{m, \psi_\nu^*} e^{-\psi_{\nu+1}^*(\beta) + l_\nu} \sum_{\substack{j \in \mathbb{Z}_+^n : \\ (0, \dots, 0) \leq j \leq \beta}} \binom{\beta}{j} j! e^{\psi_\nu^*(\beta-j)}.$$

From this we obtain that

$$|(D^\alpha g)(x)| \prod_{k=1}^n (1 + |x_k|)^{\beta_k} \leq \|g\|_{m, \psi_\nu^*} \beta! e^{-\psi_{\nu+1}^*(\beta) + l_\nu} \sum_{\substack{j \in \mathbb{Z}_+^n : \\ (0, \dots, 0) \leq j \leq \beta}} \frac{e^{\psi_\nu^*(\beta-j)}}{(\beta-j)!}.$$

Recall that by the Corollary 1 the series $\sum_{j \in \mathbb{Z}_+^n} \frac{e^{\psi_\nu^*(j)}}{j!}$ is converging. From this and the previous inequality it follows that for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq m$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

$$|(D^\alpha g)(x)| \prod_{k=1}^n (1 + |x_k|)^{\beta_k} \leq c_1 \|g\|_{m, \psi_\nu^*} \beta! e^{-\psi_{\nu+1}^*(\beta)}, \quad (13)$$

where $c_1 = e^{l_\nu} \sum_{j \in \mathbb{Z}_+^n} \frac{e^{\psi_\nu^*(j)}}{j!}$. Now let

$$f(\xi) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} g(x) e^{i\langle x, \xi \rangle} dx, \quad \xi \in \mathbb{R}^n.$$

For all $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n, \xi \in \mathbb{R}^n$ we have that

$$(i\xi)^\beta (D^\alpha f)(\xi) = \frac{(-1)^{|\beta|}}{(2\pi)^n} \int_{\mathbb{R}^n} D^\beta(g(x)(ix)^\alpha) e^{i\langle x, \xi \rangle} dx.$$

For $s = 1, \dots, n$ put $\gamma_s = \min(\beta_s, \alpha_s)$ and take $\gamma = (\gamma_1, \dots, \gamma_n)$. Then

$$(i\xi)^\beta (D^\alpha f)(\xi) = \frac{(-1)^{|\beta|}}{(2\pi)^n} \int_{\mathbb{R}^n} \sum_{j \in \mathbb{Z}_+^n: j \leq \gamma} \binom{\beta}{j} (D^{\beta-j} g)(x) (D^j (ix)^\alpha) e^{i\langle x, \xi \rangle} dx.$$

From this we have that

$$\begin{aligned} |\xi^\beta (D^\alpha f)(\xi)| &\leq \frac{1}{(2\pi)^n} \sum_{j \in \mathbb{Z}_+^n: j \leq \gamma} \binom{\beta}{j} \int_{\mathbb{R}^n} |(D^{\beta-j} g)(x)| \frac{\alpha!}{(\alpha-j)!} |x^{\alpha-j}| dx \\ &\leq \frac{1}{(2\pi)^n} \sum_{j \in \mathbb{Z}_+^n: j \leq \gamma} \frac{\binom{\beta}{j} \alpha!}{(\alpha-j)!} \int_{\mathbb{R}^n} |(D^{\beta-j} g)(x)| \prod_{k=1}^n (1 + |x_k|)^{\alpha_k - j_k + 2} \frac{dx}{\prod_{k=1}^n (1 + |x_k|)^2}. \end{aligned}$$

Using the inequality (13) and denoting the element $(2, \dots, 2) \in \mathbb{R}^n$ by ω we have that

$$|\xi^\beta (D^\alpha f)(\xi)| \leq \frac{c_1}{2^n} \|g\|_{|\beta|, \psi_\nu^*} \sum_{j \in \mathbb{Z}_+^n: j \leq \gamma} \frac{\binom{\beta}{j} \alpha!}{(\alpha-j)!} (\alpha-j+\omega)! e^{-\psi_{\nu+1}^*(\alpha-j+\omega)}.$$

Note that using the condition i_4) it is easy to verify that $\varkappa_\nu := \sup_{x \in \mathbb{R}^n} (\psi_{\nu+2}^*(x) - \psi_{\nu+1}^*(x + \omega)) < \infty$. From this and the previous inequality it follows that

$$|\xi^\beta (D^\alpha f)(\xi)| \leq c_2 \|g\|_{|\beta|, \psi_\nu^*} (\alpha + \omega)! \sum_{j \in \mathbb{Z}_+^n: j \leq \gamma} \binom{\beta}{j} e^{-\psi_{\nu+2}^*(\alpha-j)},$$

where $c_2 = \frac{c_1 e^{\nu}}{2^n}$. Using the inequality (12) we get

$$|\xi^\beta (D^\alpha f)(\xi)| \leq c_2 e^{l_{\nu+2}} \|g\|_{|\beta|, \psi_\nu^*} (\alpha + \omega)! e^{-\psi_{\nu+3}^*(\alpha)} \sum_{j \in \mathbb{Z}_+^n : j \leq \gamma} \binom{\beta}{j} e^{\psi_{\nu+2}^*(j)}.$$

From this we obtain that

$$|\xi^\beta (D^\alpha f)(\xi)| \leq c_2 e^{l_{\nu+2}} \|g\|_{|\beta|, \psi_\nu^*} (\alpha + \omega)! e^{-\psi_{\nu+3}^*(\alpha)} \beta! \sum_{j \in \mathbb{Z}_+^n : j \leq \gamma} \frac{e^{\psi_{\nu+2}^*(j)}}{j!}.$$

Take into account that for all $m_1, m_2 \in \mathbb{Z}_+$

$$(m_1 + m_2)! \leq e^{m_1 + m_2} m_1! m_2!$$

(this inequality easily follows from the inequality $(m_1 + m_2)^{m_2} \leq m_2! e^{m_1 + m_2}$).

Using this inequality we get from the preceding inequality that

$$|\xi^\beta (D^\alpha f)(\xi)| \leq c_3 e^{|\alpha|} \beta! \|g\|_{|\beta|, \psi_\nu^*} \alpha! e^{-\psi_{\nu+3}^*(\alpha)},$$

where $c_3 = c_2 2^n e^{2n + l_{\nu+2}} \sum_{j \in \mathbb{Z}_+^n} \frac{e^{\psi_{\nu+2}^*(j)}}{j!}$. From this using the inequality

$$\psi_k^*(x) - \psi_{k+1}^*(x) \geq \sum_{j=1}^n x_j \ln 2 - a_k, \quad x \in [0, \infty)^n, k \in \mathbb{N},$$

(that holds in view of the condition i_3) and Lemma 3) we obtain that

$$|\xi^\beta (D^\alpha f)(\xi)| \leq c_4 \beta! \|g\|_{|\beta|, \psi_\nu^*} \alpha! e^{-\psi_{\nu+5}^*(\alpha)},$$

where $c_4 = c_3 e^{a_{\nu+3} + a_{\nu+4}}$. So if $k \in \mathbb{Z}_+$ then from the last inequality we get that for all $\alpha \in \mathbb{Z}_+^n, \xi \in \mathbb{R}^n$

$$(1 + \|\xi\|)^k |(D^\alpha f)(\xi)| \leq c_5 \|g\|_{k, \psi_\nu^*} \alpha! e^{-\psi_{\nu+5}^*(\alpha)},$$

where $c_5 > 0$ is some positive constant depending on ν, n and k . By Theorem 2 f can be holomorphically continued (uniquely) to entire function F_f belonging to $E(\Phi)$. Obviously by construction $g = \mathcal{F}(F_f)$. The proof of Theorem 2 (see inequalities (2) and (7)) indicates that there is a constant $c_6 > 0$ (depending on ν, n and k) such that for $z \in \mathbb{C}^n$

$$(1 + \|z\|)^k |F_f(z)| \leq c_6 \|g\|_{k, \psi_\nu^*} e^{\varphi_{\nu+8}(|\operatorname{Im} z_1|, \dots, |\operatorname{Im} z_n|)}.$$

Hence, $p_{\nu+8, k}(F_f) \leq c_6 \|g\|_{k, \psi_\nu^*}$. From this estimate it follows that the inverse mapping \mathcal{F}^{-1} is continuous.

Thus, we have proved that Fourier transform establishes a topological isomorphism between the spaces $E(\Phi)$ and $G(\Psi^*)$. \square

5 A special case of Φ

5.1. Some desired properties of the Young-Fenchel transform.

Lemma 5. *Let $u \in \mathcal{B}(\mathbb{R}^n)$. Then for each $\delta > 0$*

$$\lim_{x \rightarrow \infty} \frac{u^*((1+\delta)x) - u^*(x)}{\|x\|} = +\infty.$$

Proof. Obviously, u^* takes finite values on \mathbb{R}^n . For each $x \in \mathbb{R}^n$ denote by $\xi(x)$ a point where the supremum of the function $g_x(\xi) := \langle x, \xi \rangle - u(\xi)$ over \mathbb{R}^n is attained. Note that from the equality $u^*(x) + u(\xi(x)) = \langle x, \xi(x) \rangle$ and the fact that $\lim_{x \rightarrow \infty} \frac{u^*(x)}{\|x\|} = +\infty$ (arguments of Remark 2 can be applied here) we have that $\lim_{x \rightarrow \infty} \frac{\langle x, \xi(x) \rangle}{\|x\|} = +\infty$. Now if $\delta > 0$ is arbitrary then from this and the inequality

$$u^*((1+\delta)x) - u^*(x) \geq \delta \langle x, \xi(x) \rangle, \quad x \in \mathbb{R}^n,$$

the assertion of lemma follows. \square

Lemma 6. *Let $u \in \mathcal{B}(\mathbb{R}^n)$. Then*

$$(u[e])^*(x) + (u^*[e])^*(x) \leq \sum_{\substack{1 \leq j \leq n: \\ x_j \neq 0}} (x_j \ln x_j - x_j), \quad x = (x_1, \dots, x_n) \in [0, \infty)^n \setminus \{0\};$$

$$(u[e])^*(0) + (u^*[e])^*(0) \leq 0.$$

Proof. For each $x = (x_1, \dots, x_n) \in [0, \infty)^n$ and for each $\varepsilon > 0$ there are points $t = (t_1, \dots, t_n), \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ such that

$$(u[e])^*(x) < \langle x, t \rangle - u[e](t) + \varepsilon,$$

$$(u^*[e])^*(x) < \langle x, \xi \rangle - u^*[e](\xi) + \varepsilon.$$

From this it follows that for each $\eta \in \mathbb{R}^n$

$$(u[e])^*(x) + (u^*[e])^*(x) < \langle x, t + \xi \rangle - u[e](t) - \langle (e^{\xi_1}, \dots, e^{\xi_n}), \eta \rangle + u(\eta) + 2\varepsilon.$$

Putting here $\eta = (e^{t_1}, \dots, e^{t_n})$ we obtain that

$$(u[e])^*(x) + (u^*[e])^*(x) < \sum_{j=1}^n (x_j(t_j + \xi_j) - e^{\xi_j + t_j}) + 2\varepsilon.$$

Consequently,

$$(u[e])^*(x) + (u^*[e])^*(x) < \sum_{j=1}^n \sup_{y_j \in \mathbb{R}} (x_j y_j - e^{y_j}) + 2\varepsilon.$$

From this we get that

$$(u[e])^*(x) + (u^*[e])^*(x) < \sum_{\substack{1 \leq j \leq n: \\ x_j \neq 0}} (x_j \ln x_j - x_j) + 2\varepsilon, \quad x \in [0, \infty)^n \setminus \{0\};$$

$$(u[e])^*(0) + (u^*[e])^*(0) < 2\varepsilon.$$

Since ε is arbitrary positive number then from the last two inequalities the assertion of Lemma follows. \square

Proposition 3. *Let $u \in \mathcal{B}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ and be convex. Then*

$$(u[e])^*(x) + (u^*[e])^*(x) = \sum_{j=1}^n (x_j \ln x_j - x_j), \quad x = (x_1, \dots, x_n) \in (0, \infty)^n.$$

Proof. Let $x = (x_1, \dots, x_n) \in (0, \infty)^n$ be arbitrary. If we show that

$$(u[e])^*(x) + (u^*[e])^*(x) \geq \sum_{j=1}^n (x_j \ln x_j - x_j),$$

then (taking into account Lemma 6) the assertion will be proved. First remark that for all $\xi, \mu \in \mathbb{R}^n$

$$(u[e])^*(x) + (u^*[e])^*(x) \geq \langle x, \xi + \mu \rangle - (u[e](\xi) + u^*[e](\mu)).$$

For an arbitrary $t = (t_1, \dots, t_n) \in (0, \infty)^n$ denote $(\ln t_1, \dots, \ln t_n)$ by $\xi(t)$ and $(\ln \frac{x_1}{t_1}, \dots, \ln \frac{x_n}{t_n})$ by $\mu(t)$ and put in the above inequality $\xi = \xi(t)$, $\mu = \mu(t)$. Then we get that

$$(u[e])^*(x) + (u^*[e])^*(x) \geq \sum_{j=1}^n x_j \ln x_j - (u[e](\xi(t)) + u^*[e](\mu(t))). \quad (14)$$

Further, note that there is a point $\zeta^* = (\zeta_1^*, \dots, \zeta_n^*) \in \mathbb{R}^n$ where the supremum of the function $g_x : \zeta \in \mathbb{R}^n \rightarrow \langle x, \zeta \rangle - u[e](\zeta)$ over \mathbb{R}^n is attained. Indeed, for each $\varepsilon > 0$ there exists a point $\zeta(\varepsilon) = (\zeta_1(\varepsilon), \dots, \zeta_n(\varepsilon)) \in \mathbb{R}^n$ such that

$$(u[e])^*(x) < \langle x, \zeta(\varepsilon) \rangle - u[e](\zeta(\varepsilon)) + \varepsilon. \quad (15)$$

Remark that there exists a positive constant C depending on x such that $\|\zeta(\varepsilon)\| \leq C$ for all $\varepsilon > 0$. Otherwise, there exists a decreasing to zero sequence $(\varepsilon_m)_{m=1}^\infty$ such that $\|\zeta(\varepsilon_m)\| \rightarrow +\infty$ as $m \rightarrow \infty$. From the growth conditions on u we can find a constant $A > 0$ such that

$$u[e](\zeta) > e^{\zeta_1} + \dots + e^{\zeta_n} - A, \quad \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{R}^n.$$

Then from this and the inequality (15) we obtain that

$$(u[e])^*(x) < \langle x, \zeta(\varepsilon_m) \rangle - e^{\zeta_1(\varepsilon_m)} - \dots - e^{\zeta_n(\varepsilon_m)} + A + \varepsilon_m, \quad m \in \mathbb{N}.$$

From this inequality it follows that $(u[e])^*(x) = -\infty$. But it contradicts to the fact that $(u[e])^*(x) > -\infty$ (see Remark 2). Thus, we have shown that there exists a constant $C > 0$ (depending on x) such that $\|\zeta_\varepsilon\| \leq C$ for all $\varepsilon > 0$. Then using the Bolzano-Weierstrass theorem we can extract a sequence $(\zeta(\varepsilon_j))_{j=1}^\infty$ converging to some point of \mathbb{R}^n . Denote this point by ζ^* . Now from (15) we get that $(u[e])^*(x) \leq \langle x, \zeta^* \rangle - u[e](\zeta^*)$. From the other hand $(u[e])^*(x) \geq \langle x, \zeta \rangle - u[e](\zeta)$ for each $\zeta \in \mathbb{R}^n$. Hence, $(u[e])^*(x) = \langle x, \zeta^* \rangle - u[e](\zeta^*)$. Thus, ζ^* is the point where the supremum of the function g_x over \mathbb{R}^n is attained. Obviously, $x_j = e^{\zeta_j^*}(D_j u)(e^{\zeta_1^*}, \dots, e^{\zeta_n^*})$ ($j = 1, \dots, n$). Next, define a point $t^* = (t_1^*, \dots, t_n^*) \in (0, \infty)^n$ by the rule $t_j^* = e^{\zeta_j^*}$. Then

$$t_j^*(D_j u)(t^*) = x_j, \quad j = 1, \dots, n. \quad (16)$$

Define the function $U_x : \eta \in \mathbb{R}^n \rightarrow u[e](\xi(t^*)) + \langle e^{\mu(t^*)}, \eta \rangle - u(\eta) - \sum_{j=1}^n x_j$. In

other words, $U_x(\eta) = u(t^*) + \sum_{j=1}^n \frac{x_j \eta_j}{t_j^*} - u(\eta) - \sum_{j=1}^n x_j$, $\eta \in \mathbb{R}^n$. If $\eta \neq t^*$ then using Taylor's formula we have that for some $\tau \in \mathbb{R}^n$ (depending on η)

$$U_x(\eta) = - \sum_{|\alpha|=1} (D^\alpha u)(t^*)(\eta - t^*)^\alpha + \sum_{j=1}^n x_j \frac{\eta_j - t_j^*}{t_j^*} - \frac{1}{2} \sum_{|\alpha|=2} (D^\alpha u)(\tau)(\eta - t^*)^\alpha.$$

Taking into account (16) we get that

$$U_x(\eta) = -\frac{1}{2} \sum_{|\alpha|=2} (D^\alpha u)(\tau)(\eta - t^*)^\alpha.$$

Since u is convex then from this equality it follows that $U_x(\eta) \leq 0$. Also notice that $U_x(t^*) = 0$. Thus, $U_x(\eta) \leq 0$ for all $\eta \in \mathbb{R}^n$. From this it follows that $u[e](\xi(t^*)) + u^*[e](\mu(t^*)) \leq \sum_{j=1}^n x_j$. From the other hand for each $t \in (0, \infty)^n$ we have that

$$u[e](\xi(t)) + u^*[e](\mu(t)) = u(t) + u^* \left(\frac{x_1}{t_1}, \dots, \frac{x_n}{t_n} \right) \geq \sum_{j=1}^n x_j.$$

Thus, $u[e](\xi(t^*)) + u^*[e](\mu(t^*)) = \sum_{j=1}^n x_j$. From this and (14) the desired inequality then follows. \square

Proposition 4. *Let $u \in \mathcal{A}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ and be convex. Then*

$$(u[e])^*(x) + (u^*[e])^*(x) = \sum_{\substack{1 \leq j \leq n: \\ x_j \neq 0}} (x_j \ln x_j - x_j), \quad x = (x_1, \dots, x_n) \in [0, \infty)^n \setminus \{0\};$$

$$(u[e])^*(0) + (u^*[e])^*(0) = 0.$$

Proof. If $x \in (0, \infty)^n$ then the assertion follows from Proposition 3. Now let $x = (x_1, \dots, x_n)$ belongs to the boundary of $[0, \infty)^n$ and $x \neq 0$. Assume for simplicity that the first k ($1 \leq k \leq n-1$) coordinates of x are positive and other coordinates are equal to zero. For all $\xi = (\xi_1, \dots, \xi_n), \mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$ we have that

$$(u[e])^*(x) + (u^*[e])^*(x) \geq \sum_{j=1}^k x_j (\xi_j + \mu_j) - (u(e^{\xi_1}, \dots, e^{\xi_n}) + u^*(e^{\mu_1}, \dots, e^{\mu_n})).$$

From this inequality we get that

$$(u[e])^*(x) + (u^*[e])^*(x) \geq \sum_{j=1}^k x_j (\xi_j + \mu_j) - (u(e^{\xi_1}, \dots, e^{\xi_k}, 0, \dots, 0) + u^*(e^{\mu_1}, \dots, e^{\mu_k}, 0, \dots, 0)).$$

Let $\theta = (\theta_1, \dots, \theta_k) \in (0, \infty)^k$ be arbitrary. Putting in the above inequality $\xi_j = \ln \theta_j, \mu_j = \ln \frac{x_j}{\theta_j}$ ($j = 1, \dots, k$) we obtain that

$$(u[e])^*(x) + (u^*[e])^*(x) \geq \sum_{j=1}^k x_j \ln x_j - (u(\theta_1, \dots, \theta_k, 0, \dots, 0) + u^*(\frac{x_1}{\theta_1}, \dots, \frac{x_k}{\theta_k}, 0, \dots, 0)). \quad (17)$$

Denote the point $(x_1, \dots, x_k) \in \mathbb{R}^k$ by \check{x} and define functions u_k and $\mathcal{U}_{\check{x}}$ on \mathbb{R}^k by the rules:

$$u_k : \lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k \rightarrow u(\lambda_1, \dots, \lambda_k, 0, \dots, 0);$$

$$\mathcal{U}_{\check{x}} : \lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k \rightarrow \langle \check{x}, \lambda \rangle - u_k(e^{\lambda_1}, \dots, e^{\lambda_k}).$$

Repeating the same steps shown before in Proposition 3, we can find a point $\lambda^* = (\lambda_1^*, \dots, \lambda_k^*) \in \mathbb{R}^k$ where the supremum of the function $\mathcal{U}_{\vec{x}}$ over \mathbb{R}^k is attained. It is clear that $x_j = e^{\lambda_j^*}(D_j u_k)(e^{\lambda_1^*}, \dots, e^{\lambda_k^*})$ ($j = 1, \dots, k$). Define a point $\theta^* = (\theta_1^*, \dots, \theta_k^*) \in (0, \infty)^k$ by the rule $\theta_j^* = e^{\lambda_j^*}$ ($j = 1, \dots, k$). Then $\theta_j^*(D_j u_k)(\theta^*) = x_j$, $j = 1, \dots, k$. Using similar computations as in Proposition 3 we obtain that $u_k(\theta_1^*, \dots, \theta_k^*) + u_k^*\left(\frac{x_1}{\theta_1^*}, \dots, \frac{x_k}{\theta_k^*}\right) = \sum_{j=1}^k x_j$. Now using that $u \in \mathcal{A}(\mathbb{R}^n)$ we notice that

$$\begin{aligned} u^*\left(\frac{x_1}{\theta_1}, \dots, \frac{x_k}{\theta_k}, 0, \dots, 0\right) &= \sup_{v=(v_1, \dots, v_k, \dots, v_n) \in \mathbb{R}^n} \left(\frac{x_1 v_1}{\theta_1} + \dots + \frac{x_k v_k}{\theta_k} - u(v) \right) \\ &= \sup_{(v_1, \dots, v_k) \in \mathbb{R}^k} \left(\frac{x_1 v_1}{\theta_1} + \dots + \frac{x_k v_k}{\theta_k} - u(v_1, \dots, v_k, 0, \dots, 0) \right) \\ &= u_k^*\left(\frac{x_1}{\theta_1}, \dots, \frac{x_k}{\theta_k}\right). \end{aligned}$$

Thus, $u(\theta_1^*, \dots, \theta_k^*, 0, \dots, 0) + u^*\left(\frac{x_1}{\theta_1^*}, \dots, \frac{x_k}{\theta_k^*}, 0, \dots, 0\right) = \sum_{j=1}^k x_j$. Finally, taking into account the inequality (17) we obtain that

$$(u[e])^*(x) + (u^*[e])^*(x) \geq \sum_{j=1}^k (x_j \ln x_j - x_j).$$

From this and the assertion of Lemma 6 the desired equality follows.

If $x = 0$ then $(u[e])^*(0) = -u(0)$, $(u^*[e])^*(0) = -\inf_{\xi \in \mathbb{R}^n} u^*[e](\xi) = -u^*(0) = u(0)$. Hence, $(u[e])^*(0) + (u^*[e])^*(0) = 0$. \square

Corollary 2. *Let $u \in \mathcal{A}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$ and be convex. Then*

$$(u[e])^*(x) + (u^*[e])^*(x) \geq \sum_{j=1}^n (x_j \ln(x_j + 1) - x_j) - n, \quad x = (x_1, \dots, x_n) \in [0, \infty)^n.$$

Notice that Propositions 3 and 4 are related to the following result obtained by S.V. Popenov (see Lemma 4 in [13]): let $u \in \mathcal{A}(\mathbb{R})$ be a convex function such that $\lim_{x \rightarrow 0} \frac{u(x)}{x} = 0$, then

$$(u[e])^*(x) + (u^*[e])^*(x) = x \ln x - x, \quad x > 0,$$

$$(u(e))^*(0) + (u^*(e))^*(0) = 0.$$

5.2. Description of $E(\Phi)$ by a system of weighted C^∞ -functions.

Choose a non-negative even function $\chi \in C_0^\infty(\mathbb{R})$ with $\text{supp } \chi$ in $(-1, 1)$ and $\int_{\mathbb{R}} \chi(\xi) d\xi = 1$. Define a function ω on \mathbb{R}^n by the rule: $\omega(x_1, \dots, x_n) = \chi(x_1) \cdots \chi(x_n)$. For each $m \in \mathbb{N}$ let

$$\varphi_{m,1}(x) = \int_{\mathbb{R}^n} \varphi_m(x + \xi) \omega(\xi) d\lambda_n(\xi), \quad x \in \mathbb{R}^n.$$

Here $d\lambda_n$ is the n -dimensional Lebesgue measure. The regularity properties of convolution ensures that $\varphi_{m,1} \in C^\infty(\mathbb{R}^n)$ and $\varphi_{m,1}(x_1, \dots, x_n) = \varphi_{m,1}(|x_1|, \dots, |x_n|)$ for $(x_1, \dots, x_n) \in \mathbb{R}^n$. Using convexity of φ_m we have that

$$\varphi_m(x) \leq \varphi_{m,1}(x), \quad x \in \mathbb{R}^n. \quad (18)$$

From this it follows that $\lim_{x \rightarrow \infty} \frac{\varphi_{m,1}(x)}{\|x\|} = +\infty$. Using convexity of φ_m and since $\varphi_m|_{[0,\infty)^n}$ is not decreasing in each variable it is not difficult to show that $\varphi_m|_{[0,\infty)^n}$ is nondecreasing in each variable. Thus, the family $\Phi_1 = \{\varphi_{m,1}\}_{m=1}^\infty$ is in $\mathcal{A}(\mathbb{R}^n)$.

It is trivial to verify that for each $m \in \mathbb{N}$ and each $A > 0$ there exists a constant $s_{m,A} > 0$ such that

$$\varphi_{m,1}(x) + A \ln(1 + \|x\|) \leq \varphi_{m+1,1}(x) + s_{m,A}, \quad x \in \mathbb{R}^n. \quad (19)$$

Further, for $x = (x_1, \dots, x_n) \in [0, \infty)^n, \zeta = (\zeta_1, \dots, \zeta_n) \in [0, 1]^n$ we have that

$$\begin{aligned} \varphi_{m,1}(x + \zeta) &= \int_{\mathbb{R}^n} \varphi_m(x + \zeta + \xi) \omega(\xi) d\lambda_n(\xi) \\ &= \int_{\mathbb{R}^n} \varphi_m(|x_1 + \zeta_1 + \xi_1|, \dots, |x_n + \zeta_n + \xi_n|) \omega(\xi) d\lambda_n(\xi). \end{aligned}$$

As $\varphi_m|_{[0,\infty)^n}$ is nondecreasing in each variable then

$$\varphi_{m,1}(x + \zeta) \leq \int_{\mathbb{R}^n} \varphi_m((x_1 + \xi_1) + |\zeta_1|, \dots, (x_n + \xi_n) + |\zeta_n|) \omega(\xi) d\lambda_n(\xi).$$

Now using the condition i_2) on Φ we have that

$$\begin{aligned} \varphi_{m,1}(x + \zeta) &\leq \int_{\mathbb{R}^n} (\varphi_{m+1}(x_1 + \xi_1, \dots, x_n + \xi_n) + K_m) \omega(\xi) d\lambda_n(\xi) \\ &= \varphi_{m+1,1}(x) + K_m. \end{aligned} \quad (20)$$

Next, for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we have that

$$\varphi_{m,1}(2x) = \int_{\mathbb{R}^n} \varphi_m(|2x_1 + \xi_1|, \dots, |2x_n + \xi_n|) \omega(\xi) d\lambda_n(\xi).$$

From this using nondecreasity in each variable of $\varphi_m|_{[0,\infty)^n}$ we have that

$$\varphi_{m,1}(2x) \leq \int_{\mathbb{R}^n} \varphi_m(|2x_1| + |\xi_1|, \dots, |2x_n| + |\xi_n|) \omega(\xi) d\lambda_n(\xi).$$

Now due to the condition i_2) on Φ we have that

$$\varphi_{m,1}(2x) \leq \int_{\mathbb{R}^n} (\varphi_{m+1}(|2x_1|, \dots, |2x_n|) + K_m) \omega(\xi) d\lambda_n(\xi) = \varphi_{m+1}(2x) + K_m.$$

Thanks to the condition i_3) on Φ we get that

$$\varphi_{m,1}(2x) \leq \varphi_{m+2}(x) + K_m + a_{m+1}. \quad (21)$$

Using the inequality (18) we obtain that

$$\varphi_{m,1}(2x) \leq \varphi_{m+2,1}(x) + K_m + a_{m+1}, \quad x \in \mathbb{R}^n. \quad (22)$$

Now introduce the family $\Phi_2 = \{\varphi_{2m,1}\}_{m=1}^\infty$. Obviously, Φ_2 is in $\mathcal{A}(\mathbb{R}^n)$. From the inequalities (19), (20) and (22) it follows that Φ_2 satisfies the condition of the form i_0), i_2) and i_3).

From the inequality (21) we have that

$$\varphi_{2m,1}(x) \leq \varphi_{2m+2}(x) + K_{2m} + a_{2m+1}, \quad x \in \mathbb{R}^n.$$

This inequality and the inequality (18) mean that $E(\Phi)$ can be described by the system Φ_2 .

5.3. Proof of Theorem 4. We may assume that functions of the family Φ belong to $C^\infty(\mathbb{R}^n)$ (see subsection 5.2). Further, note that using convexity of functions of the family Φ and the condition i_3) on Φ we have that for each $k \in \mathbb{N}$

$$2\varphi_k(x) \leq \varphi_{k+1}(x) + a_k + \varphi_k(0), \quad x \in \mathbb{R}^n.$$

This means that the condition i_4) holds in our case with $l_k = a_k + \varphi_k(0)$. From the last inequality it follows that for each $k \in \mathbb{N}$ we have that

$$2\psi_k(x) \leq \psi_{k+1}(x) + a_k + \varphi_k(0), \quad x \in \mathbb{R}^n.$$

Hence, the inequality (12) holds in our case.

Now let $\nu \in \mathbb{N}$ and $f \in G(\psi_\nu^*)$. Let $m \in \mathbb{Z}_+$ be arbitrary. For all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq m$, $\beta \in \mathbb{Z}_+^n$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with non-zero coordinates we have that

$$|(D^\alpha f)(x)| \leq \frac{\|f\|_{m, \psi_\nu^*} \beta! e^{-\psi_\nu^*(\beta)}}{\prod_{j=1}^n |x_j|^{\beta_j}}.$$

Take into account that $j! \leq \frac{(j+1)^{j+1}}{e^j}$ for all $j \in \mathbb{Z}_+$. Then

$$|(D^\alpha f)(x)| \leq \|f\|_{m, \psi_\nu^*} e^{-\psi_\nu^*(\beta)} \prod_{j=1}^n \frac{(\beta_j + 1)^{\beta_j + 1}}{(e|x_j|)^{\beta_j}}. \quad (23)$$

Our aim is to obtain a suitable estimate of $e^{-\psi_\nu^*(\beta)} \prod_{j=1}^n \frac{(\beta_j + 1)^{\beta_j + 1}}{(e|x_j|)^{\beta_j}}$ from above.

For $\beta \in \mathbb{Z}_+^n$ let $\Omega_\beta = \{\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n : \beta_j \leq \xi_j < \beta_j + 1 \ (j = 1, \dots, n)\}$. Also, for $\lambda > 0$ let $\tilde{\lambda} := \max(\lambda, 1)$. Using the inequality (12) and nondecreasity in each variable of ψ^* in $[0, \infty)^n$ we have that for $\xi \in \Omega_\beta$ and $\mu = (\mu_1, \dots, \mu_n) \in (0, \infty)^n$

$$e^{-\psi_\nu^*(\beta)} \prod_{j=1}^n \frac{(\beta_j + 1)^{\beta_j + 1}}{\mu_j^{\beta_j}} \leq e^{-\psi_{\nu+1}^*(\xi) + \psi_\nu^*(1, \dots, 1) + l_\nu} \prod_{j=1}^n \frac{\tilde{\mu}_j (\xi_j + 1)^{\xi_j + 1}}{\mu_j^{t_j}}.$$

Denoting $e^{\psi_\nu^*(1, \dots, 1) + l_\nu}$ by C_1 we rewrite the last inequality in the following form

$$e^{-\psi_\nu^*(\beta)} \prod_{j=1}^n \frac{(\beta_j + 1)^{\beta_j + 1}}{\mu_j^{\beta_j}} \leq C_1 e^{\sum_{j=1}^n (\ln \tilde{\mu}_j + (t_j + 1) \ln(t_j + 1) - t_j \ln \mu_j) - \psi_{\nu+1}^*(t)}.$$

Now using the Corollary 2 we obtain that

$$e^{-\psi_\nu^*(\beta)} \prod_{j=1}^n \frac{(\beta_j + 1)^{\beta_j + 1}}{\mu_j^{\beta_j}} \leq C_2 e^{\sum_{j=1}^n (\ln \tilde{\mu}_j + \ln(t_j + 1) - t_j \ln \mu_j + t_j) + (\varphi_{\nu+1}^*[e])^*(t)},$$

where $C_2 = C_1 e^n$. Obviously, there exists a constant $C_3 > 0$ such that

$$e^{-\psi_\nu^*(\beta)} \prod_{j=1}^n \frac{(\beta_j + 1)^{\beta_j + 1}}{\mu_j^{\beta_j}} \leq C_3 e^{(\varphi_{\nu+1}^*[e])^*(t) - \sum_{j=1}^n t_j \ln \frac{\mu_j}{4} + \sum_{j=1}^n \ln \tilde{\mu}_j}.$$

From this it follows that

$$\inf_{\beta \in \mathbb{Z}_+^n} e^{-\psi_\nu^*(\beta)} \prod_{j=1}^n \frac{(\beta_j + 1)^{\beta_j + 1}}{\mu_j^{\beta_j}} \leq C_3 e^{t \inf_{\beta \in [0, \infty)^n} ((\varphi_{\nu+1}^*[e])^*(t) - \sum_{j=1}^n t_j \ln \frac{\mu_j}{4}) + \sum_{j=1}^n \ln \tilde{\mu}_j}.$$

In other words,

$$\inf_{\beta \in \mathbb{Z}_+^n} e^{-\psi_{\nu+3n}^*(\beta)} \prod_{j=1}^n \frac{(\beta_j + 1)^{\beta_j + 1}}{\mu_j^{\beta_j}} \leq C_3 e^{-\sup_{t \in [0, \infty)^n} (\sum_{j=1}^n t_j \ln \frac{\mu_j}{4} - (\varphi_{\nu+1}^*[e])^*(t)) + \sum_{j=1}^n \ln \tilde{\mu}_j}.$$

Taking into account, by Remark 2, that the function $(\varphi_{\nu+1}^*[e])^*$ takes finite values on $[0, \infty)^n$ and $(\varphi_{\nu+1}^*[e])^*(x) = +\infty$ if $x \notin [0, \infty)^n$ we can rewrite the last inequality in the following form

$$\inf_{\beta \in \mathbb{Z}_+^n} e^{-\psi_{\nu+3n}^*(\beta)} \prod_{j=1}^n \frac{(\beta_j + 1)^{\beta_j+1}}{\mu_j^{\beta_j}} \leq C_3 e^{-\sup_{t \in \mathbb{R}^n} (\sum_{j=1}^n t_j \ln \frac{\mu_j}{4} - (\varphi_{\nu+1}^*[e])^*(t)) + \sum_{j=1}^n \ln \tilde{\mu}_j}.$$

Note that by Lemma 4 the function $\varphi_{\nu+1}^*[e]$ is convex on \mathbb{R}^n with finite values (thus, $\varphi_{\nu+1}^*[e]$ is continuous on \mathbb{R}^n (see [16], Corollary 10.1.1)). Taking into account that the Young-Fenchel conjugation is involutive (see [16], Theorem 12.2) we have that

$$\sup_{t \in [0, \infty)^n} (\sum_{j=1}^n t_j \ln \frac{\mu_j}{4} - (\varphi_{\nu+1}^*[e])^*(t)) = \varphi_{\nu+1}^*[e] \left(\ln \frac{\mu_1}{4}, \dots, \ln \frac{\mu_n}{4} \right).$$

Thus,

$$\inf_{\beta \in \mathbb{Z}_+^n} e^{-\psi_{\nu}^*(\beta)} \prod_{j=1}^n \frac{(\beta_j + 1)^{\beta_j+1}}{\mu_j^{\beta_j}} \leq C_3 e^{-\varphi_{\nu+1}^*[e](\ln \frac{\mu_1}{4}, \dots, \ln \frac{\mu_n}{4}) + \sum_{j=1}^n \ln \tilde{\mu}_j}.$$

In other words,

$$\inf_{\beta \in \mathbb{Z}_+^n} e^{-\psi_{\nu}^*(\beta)} \prod_{j=1}^n \frac{(\beta_j + 1)^{\beta_j+1}}{\mu_j^{\beta_j}} \leq C_3 e^{-\varphi_{\nu+1}^*(\frac{\mu}{4}) + \sum_{j=1}^n \ln \tilde{\mu}_j}. \quad (24)$$

Note that using the condition $i_3)$ on Φ it is easy to obtain that for each $j \in \mathbb{N}$

$$\varphi_{j+1}^*(\xi) \leq \varphi_j^* \left(\frac{\xi}{2} \right) + a_j, \quad \xi \in \mathbb{R}^n. \quad (25)$$

Due to this inequality we get from (24) that

$$\inf_{\beta \in \mathbb{Z}_+^n} e^{-\psi_{\nu}^*(\beta)} \prod_{j=1}^n \frac{(\beta_j + 1)^{\beta_j+1}}{\mu_j^{\beta_j}} \leq C_4 e^{-\varphi_{\nu+3}^*(\mu) + \sum_{j=1}^n \ln \tilde{\mu}_j}, \quad (26)$$

where $C_4 = C_3 e^{a_{\nu+1} + a_{\nu+2}}$. Also note that from (25) it follows that for each $j \in \mathbb{N}$

$$\varphi_j^*(\xi) - \varphi_{j+1}^*(\xi) \geq \varphi_j^*(\xi) - \varphi_j^* \left(\frac{\xi}{2} \right) - a_j, \quad \xi \in \mathbb{R}^n.$$

From this and Lemma 5 we get that

$$\lim_{\xi \rightarrow \infty} \frac{\varphi_j^*(\xi) - \varphi_{j+1}^*(\xi)}{\|\xi\|} = +\infty.$$

Using this we obtain from (26) that

$$\inf_{\beta \in \mathbb{Z}_+^n} e^{-\psi_\nu^*(\beta)} \prod_{j=1}^n \frac{(\beta_j + 1)^{\beta_j+1}}{\mu_j^{\beta_j}} \leq C_5 e^{-\varphi_{\nu+4}^*(\mu)},$$

where C_5 is some positive number depending on ν . From this and the inequality (23) we obtain at last that for all $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with non-zero coordinates and for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq m$

$$|(D^\alpha f)(x)| \leq C_5 \|f\|_{m, \psi_\nu^*} e^{-\varphi_{\nu+4}^*(e|x_1|, \dots, e|x_n|)} \leq C_5 \|f\|_{m, \psi_\nu^*} e^{-\varphi_{\nu+4}^*(x)}.$$

Obviously, the last inequality holds for all $x \in \mathbb{R}^n$. Thus,

$$q_{m, \nu+4}(f) \leq C_5 \|f\|_{m, \psi_\nu^*}, \quad f \in G(\psi_\nu^*).$$

From this it follows that the identity mapping J acts from $G(\Psi^*)$ to $GS(\Phi^*)$ and is continuous.

We proceed to show that J is surjective. Let $f \in GS(\Phi^*)$. Then $f \in GS(\varphi_\nu^*)$ for some $\nu \in \mathbb{N}$. Fix $m \in \mathbb{Z}_+$. Consider an arbitrary point $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ with non-zero coordinates. For all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq m$ we have that

$$|(D^\alpha f)(x)| \leq q_{m, \nu}(f) e^{-\varphi_\nu^*(|x_1|, \dots, |x_n|)}.$$

In other words,

$$|(D^\alpha f)(x)| \leq q_{m, \nu}(f) e^{-\varphi_\nu^*[e](\ln |x_1|, \dots, \ln |x_n|)}.$$

From this we have that

$$|(D^\alpha f)(x)| \leq q_{m, \nu}(f) \exp\left(-\sup_{t=(t_1, \dots, t_n) \in [0, \infty)^n} \left(\sum_{j=1}^n t_j \ln |x_j| - (\varphi_\nu^*[e])^*(t)\right)\right).$$

Now using Proposition 4 we get that

$$|(D^\alpha f)(x)| \leq q_{m, \nu}(f) e^{-\sup_{t=(t_1, \dots, t_n) \in [0, \infty)^n} \left(\sum_{1 \leq j \leq n: t_j \neq 0} (t_j \ln(e|x_j|) - t_j \ln t_j) + \psi_\nu^*(t)\right)}.$$

Consequently, if $\beta \in \mathbb{Z}_+^n$ then

$$|(D^\alpha f)(x)| \leq q_{m, \nu}(f) e^{-\psi_\nu^*(\beta)} \prod_{1 \leq j \leq n: \beta_j \neq 0} \frac{\beta_j^{\beta_j}}{(e|x_j|)^{\beta_j}}$$

From this we finally obtain that for all $x \in \mathbb{R}^n$ with non-zero coordinates

$$|x^\beta (D^\alpha f)(x)| \leq q_{m, \nu}(f) \beta! e^{-\psi_\nu^*(\beta)}, \quad \beta \in \mathbb{Z}_+^n, |\alpha| \leq m.$$

Clearly, this inequality holds for all $x \in \mathbb{R}^n$. Thus,

$$\|f\|_{m,\psi_\nu^*} \leq q_{m,\nu}(f).$$

Since here $m \in \mathbb{Z}_+$ is arbitrary then $f \in G(\psi_\nu^*)$. Hence, $f \in G(\Psi^*)$. Also from the last inequality it follows that the mapping J^{-1} is continuous. Thus, the spaces $G(\Psi^*)$ and $GS(\Phi^*)$ coincide. \square

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